

y-DEFORMED BPS Dp- BRANES ON A SURFACE IN A CALABI-YAU THREEFOLD

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ABSTRACT

Using y-deformed algebraic geometric techniques the y-deformed Mukay vector of RR-charges of the y-deformed BPS Dp-branes localized on a surface in a Calabi-Yau threefold. The formulae that are obtained here are generalizations of the formulae of the fourth section of the preprint hep-th/0007243

1 Introduction:y-deformed BPS Dp-branes on a Calabi-Yau threefold

A BPS D-brane on a Calabi-Yau threefold X can be represented using a coherent \mathcal{O}_X -module G . The RR charge of G is given by the Mukai vector[1]:

$$v_X(G) = ch(G)\sqrt{Todd(T_X)} \in H_{2*}(X; \mathbb{Q}) := \bigoplus_{i=0}^3 H_{2i}(X; \mathbb{Q})$$

where $ch(G) = \sum_{i=0}^3 ch_i(G)$ is the Chern character with $ch_i(G) \in H_{6-2i}(X; \mathbb{Q})$, which can be computed by the homology-cohomology duality[1]: always one can to have a resolution of G by locally free sheaves (V_*) , in such way that one can to set that $ch(G) := \sum_{i=0}^3 (-1)^i ch(V_i)$, and these result does not depend on the choice of the resolution. Finally $Todd(T_X) = [X] + \frac{c_1[X]}{2} + \frac{c_2[X] + c_1[X]^2}{12} + \frac{c_2[X]c_1[X]}{24}$. Now when X is a Calabi-Yau threefold one has $c_1[X] = 0$ and then one obtains: $Todd(T_X) = [X] + \frac{c_2[X]}{12}$. From these the effect of the square root of the Todd Class on the RR charges, is to say the geometric version of the Witten effect is given by:

$$\sqrt{Todd(T_X)} = [X] + \frac{c_2[X]}{24}$$

For the investigation of the topological aspects of D-branes is of the great importance to obtain several basic invariants of BPS D-Branes. One of these invariants is the RR charge of the D-brane. Other invariant is the intersection

form on D-branes on X [1]. This invariant for intersections of two Dp-branes is obtained by multiplication of the Mukay vectors of RR charges corresponding to the intersecting Dp-branes and is given by: [1]

$$\begin{aligned} I_X(G_1, G_2) &= [v_X(G_1)^\vee \cdot v_X(G_2)]_X = \\ &= [(ch(G_1)\sqrt{Todd(T_X)})^\vee \cdot ch(G_2)\sqrt{Todd(T_X)}]_X = \\ &= [ch(G_1)^\vee \cdot ch(G_2) Todd(T_X)]_X \end{aligned}$$

where $[...]_X$ evaluates the degree of $H_0(X; Q) \cong Q$ component, and v^\vee flips the sign of $H_0(X) \oplus H_4(X)$ part of the Mukay vector v . In particular, if G itself is locally free, then $ch(G)^\vee = ch(G^\vee)$, where $G^\vee = Hom_X(G, 0_X)$ is the dual sheaf. Finally is easy to check that: $I_X(G_1, G_2) = -I_X(G_2, G_1)$. On other hand the invariant of intersection between D-branes is an application of the Hirzebruch-Riemann-Roch and for then you can write[1]

$$I_X(G_1, G_2) = \sum_{i=0}^3 (-1)^i dim Ext_X^i(G_1, G_2)$$

For this reason the skew-symmetric property $I_X(G_1, G_2) = -I_X(G_2, G_1)$ of the intersection form I_X for the intersection of two Dp-branes may be attributed to the Serre duality: $Ext_X^i(G_1, G_2) \cong Ext_X^{3-i}(G_1, G_2)^\vee$ [1]. Another interesting comentary is that from the integrality theorems for diferential and complex manifolds the formula H.R.R. is an integer and this assures that I_X takes values in \mathbb{Z} . [1],[2].

Now the result that this work presents is about the y-deformed Dp-branes on a Calabi Yau threefold. A y-deformed BPS Dp-brane on a Calabi-yau X can be represented by a y-deformed O_X – modulo G . The y-deformed RR charge of G is given by the y-deformed Mukai vector:

$$\begin{aligned} v_{X \rightarrow Y}(G) &= ch_Y(G) \sqrt{\chi_Y(T_X)} \in (H_{2*}(X; Q) \otimes Q[y]) := \\ &= \bigoplus_{i=0}^3 (H_{2i}(X; Q) \otimes Q[y]) \end{aligned}$$

where χ_y is the y-chi-genus which is a generalization of the Todd class [2,3] and $ch_y(G)$ is the y-deformed Chern Character. the total Chern Class for T_X has the following sumarization:

$$c(T_X) = \sum_{j=0}^3 c_j(T_X)$$

also, the total Chern Class for the such bundle has the following factorization:

$$c(T_X) = \prod_{i=1}^3 (1 + x_i)$$

The CHI-y- genus for T_X has the following formal factorisation:

$$\chi_y(T_X) = \prod_{i=1}^3 \frac{(1+y\exp(-(y+1)x_i))x_i}{1-\exp(-(y+1)x_i)}$$

The CHI-y- genus for T_X has the following formal sumarisation in terms of the y-deformed Todd polynomials which are formed from the corresponding Chern classes and from the polynomials on y :

$$\chi_y(T_X) = \sum_{j=0}^{\infty} T_j(c_1(T_X), \dots, c_j(T_X), y)$$

The y-Todd polynomials are given by:

$$T_0(c_0(T_X), y) = T_0(1, y) = 1$$

$$T_1(c_1(T_X), y) = \frac{(1-y)c_1(T_X)}{2}$$

$$T_2(c_1(T_X), c_2(T_X), y) = \frac{(y+1)^2 c_1(T_X)^2 + (y^2 - 10y + 1)c_2(T_X)}{12}$$

$$T_3(c_1(T_X), c_2(T_X), c_3(T_X), y) = \frac{- (y+1)^2 (y-1)c_1(T_X)c_2(T_X) + 12y(y-1)c_3(T_X)}{24}$$

Then one has:

$$\chi_y(T_X) = 1 + \frac{(1-y)c_1(T_X)}{2} + \frac{(y+1)^2 c_1(T_X)^2 + (y^2 - 10y + 1)c_2(T_X)}{12} + \frac{- (y+1)^2 (y-1)c_1(T_X)c_2(T_X) + 12y(y-1)c_3(T_X)}{24}$$

When X is a Calabi-Yau threefold then the chi-y-genus is given by

$$\chi_y(T_X) = 1 + \frac{(y^2-10y+1)c_2(T_X)}{12} + \frac{12y(y-1)c_3(T_X)}{24}$$

From this one can write the following formula for the y-deformed geometric version of the Witten effect:

$$\sqrt{\chi_y(T_X)} = [X] + \frac{(y^2-10y+1)c_2[X]}{24} + \frac{y(y-1)c_3[X]}{4}$$

when $y=0$ one obtains the usual Witten effect:

$$\sqrt{\chi_0(T_X)} = [X] + \frac{(0^2-0+1)c_2[X]}{24} + \frac{0(0-1)c_3[X]}{4} = [X] + \frac{c_2[X]}{24}$$

For the other hand the y-deformed Chern Character $ch_y(G)$ is given by: $ch_y(G) = \sum_{i=0}^3 ch_{i,y}(G)$ with $ch_{i,y}(G) \in (H_{6-2i}(X; Q) \otimes Q[y])$, which can be computed using y-deformed homology-cohomology duality: always one can have a y-deformed resolution of G by y-deformed locally free sheaves (V_*) , in such way that one can set that $ch_y(G) := \sum_{i=0}^3 (-1)^i ch_y(V_i)$, and these result does not depend on the choice of the y-deformed resolution. The total Chern Class for G has the following sumarization:

$$c(G) = \sum_{j=0}^q c_j(G)$$

also, the total Chern Class for G has the following factorization:

$$c(G) = \prod_{i=1}^q (1 + z_i)$$

The total Chern character of G is defined by:

$$ch(G) = \sum_{j=1}^q e^{z_j}$$

The total y-deformed Chern character for G has the following sumarization:

$$ch_y(G) = \sum_{j=1}^q e^{(1+y)z_i}$$

The total y-deformed Chern character for G has the following expansion in terms of the Chern class of G and polynomials for y:

$$ch_y(G) = rk(G) + (y+1)c_1(G) + (y+1)^2\left(\frac{c_1(G)^2 - c_2(G)}{2}\right) + (y+1)^3\left(\frac{c_1(G)^3 - 3c_1(G)c_2(G) + 3c_3(G)}{6}\right)$$

It is easy to see that when $y=0$, one obtains the usual expansion for the usual Chern character. For the investigation of the topological aspects of the y-deformed D-branes is of the great importance to obtain several basic y-deformed invariants of y-deformed BPS D-Branes. One of these y-deformed invariants is the y-deformed RR charge of the y-deformed D-brane. Other y-deformed invariant is the y-deformed intersection form on y-deformed D-branes on X. This y-deformed invariant for intersections of two y-deformed Dp-branes is obtained by multiplication of the y-deformed Mukay vectors of the y-deformed RR charges corresponding to the intersecting y-deformed Dp-branes and is given by:

$$I_{X,y}(G_1, G_2) = [v_{X,y}(G_1)^\vee \cdot v_{X,y}(G_2)]_X =$$

$$[(ch(G_1)\sqrt{\chi_y(T_X)})^\vee \cdot ch(G_2)\sqrt{\chi_y(T_X)}]_X = [ch(G_1)^\vee \cdot ch(G_2)\chi_y(T_X)]_X$$

where $[...]_{X,y}$ evaluates the degree of $(H_0(X; Q) \otimes Q[y]) \cong (Q \otimes Q[y])$ component, and v^\vee flips the sign of $(H_0(X) \otimes Q[y]) \oplus (H_4(X) \otimes Q[y])$ y-deformed part of the y-deformed Mukay vector v . In particular, if G itself is locally free, then $ch_y(G)^\vee = ch_y(G^\vee)$, where $G^\vee = Hom_X(G, 0_X)$ is the y-deformed dual sheaf. Finally is easy to check that: $I_{X,y}(G_1, G_2) = -I_{X,y}(G_2, G_1)$.

On other hand the y-deformed invariant of intersection between y-deformed D-branes is an application of the y-deformed Hirzebruch-Riemann-Roch and for then you can write:

$$I_{X/y}(G_1, G_2) = \sum_{i=0}^3 (-1)^i \dim Ext_{X/y}^i(G_1, G_2)$$

For this reason the skew-symmetric property $I_{X,y}(G_1, G_2) = -I_{X,y}(G_2, G_1)$ of the intersection form $I_{X,y}$ for the intersection of two y-deformed Dp-branes may be attributed to the y-deformed Serre duality: $Ext_{X,y}^i(G_1, G_2) \cong Ext_{X,y}^{3-i}(G_1, G_2)^\vee$. Another interesting comentary is that from the y-deformed integrality theorems for diferential and complex manifolds the y-deformed formula H.R.R. is an polynomial on y and this assures that $I_{X,y}$ takes values in $\mathbb{Q}[y]$.

Now let $J_{X,y} \in (H_4(X; R) \otimes R[y])$ be a y-deformed Kahler form on X, whis is here identified with an y-deformed R-extended ample divisor. The y-deformed classical expression of the y-deformed central charge of the y-deformed D-brane G is then given by [1]:

$$Z_{J_{X,y}}^d(G) = -[e^{-J_{X,y}} \cdot v_{X/y}(G)]_X = - \sum_{k=0}^3 \frac{(-1)^k}{k!} [J_{X/y}^k \cdot v_{X/y}(G)]_X$$

where $v_{X,y,k}$ is the $H_{2k}(X) \otimes \mathbb{Q}[y]$ component of $v_{X,y} \in (H_{2*}(X; \mathbb{Q}) \otimes \mathbb{Q}[y])$.

In such way we obtain the three y-deformed invariants: y-deformed RR charge, y-deformed central charge and y-deformed intersections pairings of two y-deformed BPS Dp-branas. With this aid of some algebraic geometry-topology techniques we can to begin the study of topological aspects of y-deformed BPS Dp-branes bounded on a projective algebraic surface in a Calabi-Yau threefold X.

2 y-deformed BPS Dp-branes localized on a surface in a Calabi-Yau threefold

Let f be an embedding of a projective algebraic surface S in a Calabi-Yau threefold X. In the limit of infinite elliptic fiber, the y-deformed BPS Dp-branes for which the y-deformed central charge remains finite are those y-deformed BPS Dp-branes which are confined to the algebraic surface S. The physical and topological propertis of the y-deformed BPS D-p-branes localized on the

algebraic surface S then depend not on the details of the global model X , but only on the intrinsic y -deformed geometry of S and its y -deformed normal bundle $N_{S,y} = N_{S|X,y}$ which is isomorphic to the y -deformed canonical line bundle $K_{S,y}$. In particular, this means that we can compute the y -deformed central charges of y -deformed BPS D-p-branes using y -deformed local mirror symmetry principle on S .

In a elementary physical configuration you have a y -deformed BPS Dp-brane sticking to S . Such y -deformed D-brane sticking to S can be described mathematically by a y -deformed $O_S - module E$. For this configuration an important y -deformed topological invariant is the y -deformed Euler number of E (the Euler y -polynomial for E) which is defined by $\chi_y(E) = \sum_{j=0}^2 (-1)^j h^j(S, E, y)$, where $h^i(S, E, y) = \dim(H^i(S, E))_y$. For to obtain the y -deformed Euler number of E or the Euler polynomial of E the first thing that one needs is the y -deformed Todd class of S or χ_y class of S :

$$\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \frac{(y+1)^2 c_1(S)^2 + (y^2 - 10y + 1)c_2(S)}{12}$$

this expansion can be written as:

$$\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S)[pt]$$

where:

$$\chi_y(O_S) = \left[\frac{(y+1)^2 c_1(S)^2 + (y^2 - 10y + 1)c_2(S)}{12} \right]_S$$

The second thing for to do is to apply the y -deformed H.R.R formula, and then one get:

$$\begin{aligned} \chi_y(E) &= [ch_y(E)\chi_y(T_S)]_S = [ch_y(E)([S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S))]_S = \\ &= [(rk(E) + (y+1)c_1(E) + (y+1)^2(\frac{c_1(E)^2 - c_2(E)}{2}))([S] + \frac{(1-y)c_1(S)}{2} + \end{aligned}$$

$$\chi_Y(\mathcal{O}_S)]_S =$$

$$rk(E)\chi_Y(\mathcal{O}_S) + [(y+1)^2(\frac{c_1(E)^2 - c_2(E)}{2}) + \frac{(y+1)(1-y)c_1(S) \cdot c_1(E)}{2}]_S$$

From the other side, there is y -deformed canonical push-forward homomorphism f_* from $H_{2*}(S; Q) \otimes Q[y]$ to $H_{2*}(X; Q) \otimes Q[y]$, which maps a y -deformed cycle on S that on X . Also, one can define the y -deformed coherent sheaf $f_!E$ on X by extending E by zero to X/S . Now using the y -deformation of the celebrated Grothendieck-Riemann-Roch formula for the embedding f of S in X , one can relate the y -deformed Chern characters of E and $f_!E$ as follows:

$$ch_y(f_!E) = f_*(ch_y(E) \frac{1}{chi_y(N_S)})$$

Multiplying the both sides of the y -deformed GRR formula by $\sqrt{\chi_y(T_X)}$, one has:

$$ch_y(f_!E) \sqrt{\chi_y(T_X)} = f_*(ch_y(E) \sqrt{\frac{chi_y(T_S)}{chi_y(N_S)}})$$

where we have used the y -deformed projection formula:

$$f_*(a \cdot f^*b) = f_*a \cdot b$$

with $a \in (H_{2*}(S; Q) \otimes Q[y])$, $b \in (H_{2*}(X; Q) \otimes Q[y])$

and $f^*chi_y(T_X) = chi_y(T_S) \cdot chi_y(N_S)$, which follows from the y -deformed short exact sequence of bundles on S : $0 \rightarrow T_S \rightarrow f^*T_X \rightarrow N_S \rightarrow 0$, combined with the multiplicative property of the chi - y -genus.

Now the y -deformed BPS Dp-brane on a Calabi-Yau threefold X is represented by G and y -deformed BPS Dp-brane sticking to S can be described by E then one has $G = f_!E$ and following formula for the y -deformed Mukai vector of the y -deformed RR charges of $G = f_!E$

$$v_{X \rightarrow Y}(f_! E) = ch_Y(f_! E) \sqrt{\chi_Y(T_X)} \in (H_{2*}(X; Q) \otimes Q[y]) := \oplus_{i=0}^3 (H_{2i}(X; Q) \otimes Q[y])$$

The you have:

$$v_{X \rightarrow Y}(f_! E) = f_*(ch_Y(E) \sqrt{\frac{ch_Y(T_S)}{ch_Y(N_S)}}) = f_*(v_{S \rightarrow Y}(E))$$

In such way the y-deformed RR charge of the y-deformed BPS Dp-brane represented by E on S regarded as a y-deformed BPS Dp-brane on X can written in the following intrinsic description (of the y-deformed RR charge on S):

$$v_{S \rightarrow Y}(E) = ch_Y(E) \sqrt{\frac{ch_Y(T_S)}{ch_Y(N_S)}} = ch_Y(E) \sqrt{\frac{ch_Y(T_S)}{ch_Y(K_S)}}$$

The y-deformed gravitational correction factor for S admits the following expansion:

$$\sqrt{\frac{ch_Y(T_S)}{ch_Y(K_S)}} = [S] + \frac{(1-y)c_1(S)}{2} + \frac{(-10y+1+y^2)c_2(S)+3(y-1)^2c_1(S)^2}{24} \in (H_{2*}(S; Q) \otimes Q[y])$$

As a simple exercise one can to compute the y-deformed RR charge of a y-deformed sheaf on S. For this let $i: C \rightarrow S$ be an embedding of a smooth genus g algebraic curve in S with the normal bundle $N_C = N_{C/S}$. Then from a lin bundle L_C on C, one obtains a y-deformed torsion sheaf $i_! L_C$ on S and $ch_y(i_! L_C)$ can be computed from the y-deformed G.R.R. formula:

$$ch_Y(i_! L_C) = i_*(ch_Y(L_C) \frac{1}{ch_Y(N_C)}) = i_*((rk(L_C) + (y+1)c_1(L_C)(1 + \frac{(y-1)c_1(N_C)}{2})) = i_*[C] + ((y+1)c_1(L_C) + \frac{(y-1)deg(N_C)}{2})[pt] = i_*[C] + ((y+1)deg(L_C) + \frac{(y-1)deg(N_C)}{2})[pt]$$

where $\deg(L) := [c_1(L)]_C$ for a line bundle on C . Then y -deformed RR charge of the y -deformed BPS Dp-brane bounded on S represented by the y -deformed O_S - module $i_! L_C$ can be computed as follows:

$$v_{S,y}(i_! L_C) = ch_y(i_! L_C) \sqrt{\frac{\chi_y(T_C)}{\chi_y(K_C)}} =$$

$$(i_*[C] + ((y+1)\deg(L_C) + \frac{(y-1)\deg(N_C)}{2})[pt])([C] + \frac{(1-y)c_1(C)}{2}) =$$

$$(i_*[C] + ((y+1)\deg(L_C) + (1-y)c_1(C))[pt]) \in \oplus (H_0(S) \otimes Q[y])$$

I now turn again to intersection pairings of the y -deformed BPS Dp-branes one has the question about the what is the most appropriate intersection for on y -deformed D-branes on S . Here we will describe only y -deformed candite.

The y -deformed candidate uses the intrinsic y -deformed Mukay vector $v_{S,y}$ and defines a y -deformed symmetric form:

$$I_{S,y}(E_1, E_2) = -[v_{S,y}(E_1)^\vee \cdot v_{S,y}(E_2)]_S = \frac{r_1 r_2 (y^2 - 10y + 1) \chi(S)}{12} + [r_1 ch_2(E_2) + r_2 ch_2(E_1) - c_1(E_1) \cdot c_1(E_2)]_S$$

where $ch(E) = r[S] + c_1(E) + ch_2(E)$, $\chi(S) = [c_2(S)]_S$ Is THE euler number, and $v_y^\vee = -v_{0,y} + v_{1,y} - v_{2,y}$ with $v_{i,y}$ being the y -deformed $(H_{2i}(S) \otimes Q[y])$ componente of the y -deformed vector v_y .

In constrast with I_X that have values in $Q[y]$ and when $y=0$ then takes values in \mathbb{Z} , now I_S also have values in $Q[y]$ but in this case when $y=0$ I_S is not \mathbb{Z} -valued in general.

3 References

- [1] hep-th/0007243
- [2] F. Hirzebruch, Topological Methods in Algebraic Geometry, 1978